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Another criterion for supersolvability of finite groups

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ABSTRACT

Let $o(G)$ be the average order of a finite group G. In this paper, we prove that if $o(G) < \frac{31}{12}$, then G is supersolvable. Moreover, we have $o(G) = \frac{31}{12}$ if and only if $G \cong A_4$. We also classify finite groups G satisfying $o(G) < \frac{31}{12}$.

1. Introduction

Given a finite group G, we denote by $\psi(G)$ the sum of element orders of G and by $o(G)$ the average order of G, that is

$$\psi(G) = \sum_{x \in G} o(x) \text{ and } o(G) = \frac{\psi(G)}{|G|}\,.$$

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In the last years there has been a growing interest in studying the properties of these functions and their relations with the structure of G (see for example [1–4], [7–9], [11,12], [15] and [19,20]).

In [11], A. Jaikin-Zapirain uses the average order to determine a lower bound for the number of conjugacy classes of a finite p-group/nilpotent group. He also suggests the following question: "*Let G be a finite (p-)group and N be a normal (abelian) subgroup of G. Is it true that $o(G) \geq o(N)^{\frac{1}{2}}$?*". Recently, E.I. Khukhro, A. Moretó and M. Zarrin proved the following result (see Theorem 1.2 of [12]):

Theorem A. *Let $c > 0$ be a real number and $p \geq \frac{3}{c}$ be a prime. Then there exists a finite p-group G with a normal abelian subgroup N such that $o(G) < o(N)^c$.*

Note that Theorem A provides a negative answer to Jaikin-Zapirain's question even if we replace the exponent $\frac{1}{2}$ with any positive real number c. In the same paper [12], the authors posed the following conjecture:

Conjecture. *Let G be a finite group and suppose that $o(G) < \frac{211}{60} = o(A_5)$. Then G is solvable.*

This has been confirmed by M. Herzog, P. Longobardi and M. Maj [9].

Theorem B. *Let G be a finite group. If $o(G) < \frac{211}{60}$, then G is solvable. Moreover, if G is a non-solvable finite group, we have $o(G) = \frac{211}{60}$ if and only if $G \cong A_5$.*

Inspired by these results, we came up with the following new criterion for supersolvability of finite groups.

Theorem 1.1. *Let G be a finite group. If $o(G) < \frac{31}{12}$, then G is supersolvable. Moreover, we have $o(G) = \frac{31}{12}$ if and only if $G \cong A_4$.*

Theorem 1.1 also leads to a classification of finite groups G satisfying $o(G) < \frac{31}{12}$.

Theorem 1.2. *Let G be a finite group with $o(G) < \frac{31}{12}$. Then one of the following statements holds:*

a) $G \cong C_3$;
b) $G \cong C_2^n$ *is an elementary abelian* 2*-group;*
c) $G \cong D(C_4 \times C_2^n)$ *is a generalized dihedral group;*
d) $G \cong K \rtimes H$ *is a Frobenius group whose kernel K is an elementary abelian* 3*-group and the complement H is cyclic of order* 2*; moreover, in this case we have*

$$o(G) = \frac{5 \cdot 3^m - 2}{2 \cdot 3^m},$$

where $|K| = 3^m$.

For the proof of our results, we need the following two theorems. Recall that a *just non-supersolvable group* is a solvable group which is not supersolvable, but all of whose proper quotients are supersolvable.

Theorem C. (D.J.S. Robinson and J.S. Wilson [16]) *Let G be a finite just non-supersolvable group. Then G splits over its Fitting subgroup A and all complements of A are conjugate. Moreover, A is abelian and noncyclic, $Q = G/A$ is supersolvable, and A is faithful and simple as a Q-module.*

Conversely, any extension by a finite supersolvable group Q of a faithful simple Q-module which is not $\mathbb{Z}$-cyclic is a finite just non-supersolvable group.

Theorem D. *Let G be a finite group, $\pi_e(G)$ be the set of element orders of G and $n_d(G)$ ($s_d(G)$) be the number of elements (subgroups) of order d in G, $\forall\, d \in \mathbb{N}$. Then the following statements hold:*

1) (T.J. Laffey [13]) *If p is a prime divisor of $|G|$ and G is not a p-group, then*

$$n_p(G) \leq \frac{p}{p+1}\,|G| - 1.$$

2) (T.J. Laffey [14]) *If G is a 3-group and* $\exp(G) \neq 3$, *then*

$$n_3(G) \leq \frac{7}{9}\,|G| - 1.$$

3) (C.T.C. Wall [21]) *If $n_2(G) > \frac{1}{2}\,|G| - 1$, then one of the following holds:*
   a) *$G \cong D(A)$ is a generalized dihedral group, where A is abelian;*
   b) *$G \cong D_8 \times D_8 \times E$, where E is an elementary abelian 2-group;*
   c) *$G \cong H(r) \times E$, where $H(r)$ is a central product of r copies of D_8 and E is an elementary abelian 2-group;*
   d) *$G \cong S(r) \times E$, where $S(r) = \langle x_1, y_1, ..., x_r, y_r, z \mid x_i^2 = y_i^2 = z^2 = 1$, all pairs of generators commute except $[z, x_i] = x_i y_i\rangle$ and E is an elementary abelian 2-group.*
4) (R. Brandl and W. Shi [5]) *If $\pi_e(G) = \{1, 2, 3\}$, then $G \cong K \rtimes H$ is a Frobenius group with kernel K and complement H, where either $K \cong C_3^m$ and $H \cong C_2$, or $K \cong C_2^{2m}$ and $H \cong C_3$.*
5) (M. Tărnăuceanu [18]) *If G is a 2-group of order 2^n and* $\exp(G) \neq 2$, *then*

$$s_{2^k}(G) \leq s_{2^k}(D_8 \times C_2^{n-3}),\ \forall\, k = 0, 1, \ldots, n.$$

We will also use the next two basic properties of the function $o(G)$:

- it is multiplicative, that is if $(G_i)_{i=\overline{1,k}}$ are finite groups of coprime orders, then

$$o(\prod_{i=1}^{k} G_i) = \prod_{i=1}^{k} o(G_i); \tag{1}$$

- if G is a finite group and X is a non-trivial normal subgroup of G, then

$$o\left(\frac{G}{X}\right) < o(G) \tag{2}$$

(see Lemma 3.1(2) of [9]).

Most of our notation is standard and will usually not be repeated here. Elementary notions and results on groups can be found in [10,17].

2. Proofs of the main results

We start with the following easy but important lemma.

Lemma 2.1. *Let G be a finite group, $1 = d_1 < d_2 < \ldots < d_r$ be the element orders of G and $n_{d_i}(G)$ be the number of elements of order d_i in G, $\forall\, i = 1, 2, \ldots, r$. Assume that $r \geq 3$ and take a positive integer s with $3 \leq s \leq r$. If $o(G) < c$, where $c > 0$ is a real number, then*

$$n_{d_{s-1}}(G) > \frac{d_s - c}{d_s - d_{s-1}}\,|G| - \sum_{i=1}^{s-2} \frac{d_s - d_i}{d_s - d_{s-1}}\, n_{d_i}(G). \tag{3}$$

Proof. We have $|G| = 1+n_{d_2}(G)+\ldots+n_{d_s}(G)+\ldots+n_{d_r}(G)$, so we deduce that

$$\begin{aligned}
\psi(G) &= 1+d_2 n_{d_2}(G)+\ldots+d_s n_{d_s}(G)+\ldots+d_r n_{d_r}(G)\\
&\geq 1+d_2 n_{d_2}(G)+\ldots+d_{s-1} n_{d_{s-1}}(G)+d_s(n_{d_s}(G)+\ldots+n_{d_r})(G)\\
&= 1+d_2 n_{d_2}(G)+\ldots+d_{s-1} n_{d_{s-1}}(G)+d_s(|G|-1-n_{d_2}(G)-\ldots-n_{d_{s-1}}(G)).
\end{aligned}$$

Since $o(G) < c$, it follows that $\psi(G) < c|G|$. Therefore we have

$$c|G| > 1+d_2 n_{d_2}(G)+\ldots+d_{s-1} n_{d_{s-1}}(G)+d_s(|G|-1-n_{d_2}(G)-\ldots-n_{d_{s-1}}(G)).$$

Clearly, the last inequality is equivalent to (3), completing the proof. $\square$

Our second lemma collects information about some particular classes of finite groups G satisfying $o(G) < \frac{31}{12}$.

Lemma 2.2. *Given a finite group G such that $o(G) < \frac{31}{12}$, the following statements hold:*

a) *If $|G|$ is odd, then $G \cong C_3$.*
b) *If G is abelian, then either $G \cong C_3$ or G is an elementary abelian 2-group.*
c) *If G is a non-abelian 2-group, then $G \cong D(C_4 \times E)$ is a generalized dihedral group, where E is elementary abelian.*
d) *If G is supersolvable of even order but not a 2-group, then $G \cong K \rtimes H$ is a Frobenius group whose kernel K is an elementary abelian 3-group and the complement H is cyclic of order 2; moreover, in this case we have*

$$o(G) = \frac{5 \cdot 3^m - 2}{2 \cdot 3^m},$$

where $|K| = 3^m$.

Proof. a) It suffices to observe that if $|G| \geq 5$, then

$$o(G) \geq \frac{1 + 3(|G| - 1)}{|G|} > \frac{31}{12}.$$

b) Let $G = G_1 \times G_2 \times \cdots \times G_k$, where G_i is an abelian p_i-group, $\forall\, i = 1, 2, \ldots, k$. By using (1), it follows that if $k \geq 2$, then

$$o(G) = \prod_{i=1}^{k} o(G_i) > \frac{31}{12}.$$

So, we can assume that $k = 1$, i.e. G is an abelian p-group. For $p \geq 3$ we get $G \cong C_3$ from a), while for $p = 2$ and $\exp(G) \neq 2$ we have

$$n_2(G) > \frac{4 - \frac{31}{12}}{4 - 2}\,|G| - \frac{4 - 1}{4 - 2} = \frac{17}{24}\,|G| - \frac{3}{2}$$

by (3). Then

$$|\Omega_1(G)| = n_2(G) + 1 > \frac{17}{24}\,|G| - \frac{1}{2} > \frac{1}{2}\,|G|.$$

Since $\Omega_1(G) \leq G$, it follows that $\Omega_1(G) = G$, a contradiction. Thus $\exp(G) = 2$, i.e. G is an elementary abelian 2-group.

c) By applying (3) for $s = 3$, we get

$$n_2(G) > \frac{17}{24}\,|G| - \frac{3}{2} > \frac{1}{2}\,|G| - 1.$$

Thus G is one of the groups described in Theorem D, 3). Moreover, we have $n_2(G) > \frac{2}{3}\,|G|$ if and only if $G \cong D(C_4 \times E)$ where E is elementary abelian, as it is shown in Corollary 1.4 of [6].

If $|G| > 36$, then

$$n_2(G) > \frac{17}{24}\,|G| - \frac{3}{2} > \frac{2}{3}\,|G|$$

and consequently $G \cong D(C_4 \times E)$ where E is elementary abelian.
If $|G| \leq 36$, then $|G| \leq 32$, implying that b) in Theorem D, 3), is not possible. If a), c) or d) holds, then, using the table in Theorem 1 of [6], we easily get $G \cong D_8 \times E$ with E elementary abelian of order ≤ 2.
Hence the conclusion follows.

d) Since G is supersolvable, we have $G \cong K \rtimes H$, where K is the characteristic subgroup consisting of all elements of odd order in G and H is a Sylow 2-subgroup of G. Also, we know that G has quotients isomorphic to C_2, which implies that $2 \mid |\frac{G}{G'}|$ and therefore $\frac{G}{G'} \cong C_2^q$, where $q \in \mathbb{N}^*$. This leads to $K \subseteq G'$. Let H_1 be a complement of K in G'. Then $G' = K \times H_1$ because G' is nilpotent. It follows that H_1 is characteristic in G and we have

$$\frac{G}{H_1} \cong K \rtimes \frac{H}{H_1} \cong K \rtimes C_2^q \text{ and } o\left(\frac{G}{H_1}\right) \leq o(G) < \frac{31}{12}\,.$$

Let $\pi(G) = \{2 = p_1, p_2, \ldots, p_r\}$ be the set of primes dividing $|G|$, where $p_1 < p_2 < \ldots < p_r$. By induction on r, we infer that G has a quotient of type $K_r \rtimes C_2^q$, where K_r is a Sylow p_r-subgroup of G. Take $K_{r1} \lhd K_r \rtimes C_2^q$ with $|K_{r1}| = p_r$. Then

$$\frac{K_r \rtimes C_2^q}{K_{r1}} \cong \frac{K_r}{K_{r1}} \rtimes C_2^q \text{ and } \left|\frac{K_r}{K_{r1}}\right| < |K_r|.$$

By repeating this process, we get that G has a quotient of type $C_{p_r} \rtimes C_2^q$. Now it is easy to see that the condition $o(C_{p_r} \rtimes C_2^q) < \frac{31}{12}$ implies $q = 1$ and $p_r = 3$. Thus H_1 is a maximal subgroup of H. Suppose that H contains a maximal subgroup $M \neq H_1$. Then KM is a normal subgroup of index 2 in G and so $\frac{G}{KM} \cong C_2$. This shows that $G' \subseteq KM$, which leads to $G' = KM$. Then $M \subseteq G'$ and therefore $H = MH_1 \subseteq G'$, a contradiction. Consequently, H_1 is the unique maximal subgroup of H, i.e. H is cyclic. Moreover, we remark that $H \cong C_2$ because

$$o(C_{2^t}) = \frac{2^{2t+1}+1}{3 \cdot 2^t} > \frac{31}{12}\,, \forall\, t \geq 2.$$

Hence $G \cong K \rtimes H$, where $|K| = 3^m$ and $H \cong C_2$.
Since the first three element orders of G are $d_1 = 1$, $d_2 = 2$ and $d_3 = 3$, by applying (3) for $s = 3$ we obtain

$$n_2(G) > \frac{5}{12}\,|G| - 2 = \frac{5}{6}\,3^m - 2.$$

Assume that $n_2(G) \neq 3^m$. Then $n_2(G) \leq 3^{m-1}$ because $n_2(G) \mid 3^m$, which implies that

$$\frac{5}{6}\,3^m - 2 < 3^{m-1}.$$

This leads to $m = 1$, i.e. $G \cong S_3$, a contradiction. Thus $n_2(G) = 3^m$. We also observe that G cannot have elements of order 6.

Assume that $\exp(K) \neq 3$. Then $d_4 = 9$ and from (3) for $s = 4$ it follows that

$$\begin{aligned} n_3(K) = n_3(G) &> \frac{77}{72}\,|G| - \frac{4}{3} - \frac{7}{6}\,3^m \\ &= \frac{35}{36}\,3^m - \frac{4}{3} \\ &> \frac{7}{9}\,3^m - 1, \end{aligned}$$

contradicting Theorem D, 2). Consequently, $\exp(K) = 3$ and so $\pi_e(G) = \{1, 2, 3\}$. Thus G is a Frobenius group with kernel $K \cong C_3^m$ and complement $H \cong C_2$ by Theorem D, 4). Also, it is clear that

$$o(G) = \frac{1 + 2 \cdot 3^m + 3(3^m - 1)}{2 \cdot 3^m} = \frac{5 \cdot 3^m - 2}{2 \cdot 3^m}\,,$$

completing the proof. □

The following consequence of Lemma 2.2 is immediate.

Corollary 2.3. *A finite supersolvable group G satisfying $o(G) < \frac{31}{12}$ is of one of the following types: C_3, C_2^n, $D(C_4 \times C_2^n)$ or $C_3^m \rtimes C_2$.*

We are now able to prove our main theorems.

Proof of Theorem 1.1. Assume the result is false and let G be a counterexample of minimal order. Then every proper quotient of G is supersolvable by (2). On the other hand, since

$$o(G) < \frac{31}{12} < \frac{211}{60}\,,$$

Theorem B implies that G is solvable. Thus G is a just non-supersolvable group and its structure is given by Theorem C. More precisely, G has a unique minimal normal subgroup $N \cong C_p^r$, where p is a prime and $r \geq 2$. We also have $\Phi(G) = 1$. Indeed, if $\Phi(G) \neq 1$, then $o(\frac{G}{\Phi(G)}) < o(G) < \frac{31}{12}$ implies that $\frac{G}{\Phi(G)}$ is supersolvable and so G itself is supersolvable, a contradiction. Since $N \nsubseteq \Phi(G)$, there is a maximal subgroup M of G such that $N \nsubseteq M$. Then $MN = G$ and $M \cap N \lhd G$. By the minimality of N, we get

$M \cap N = 1$, which shows that M is a complement of N in G. Moreover, N coincides with the Fitting subgroup of G and the supersolvable subgroup M acts faithfully on N. Also, we have

$$o(M) = o(\frac{G}{N}) < o(G) < \frac{31}{12}.$$

By Lemma 2.2, a), $|G| = p^r|M|$ is even and therefore we distinguish the following two cases.

Case 1. $|M|$ is odd

Then $M \cong C_3$ and $p = 2$, that is $G = C_2^r \rtimes C_3$. Since $r \geq 2$, we get

$$o(G) = \frac{1 + 2(2^r - 1) + 3 \cdot 2^{r+1}}{3 \cdot 2^r} = \frac{2^{r+3} - 1}{3 \cdot 2^r} \geq \frac{31}{12},$$

a contradiction.

Case 2. $|M|$ is even

We have the next two subcases.

Subcase 2.1. M is a 2-group

Let $|M| = 2^q$. Since G is not supersolvable, it is not strictly 2-closed and therefore $M \ncong C_2^q$. We also remark that p must be odd.

If $p \geq 5$, then the first three element orders of G are 1, 2 and 4. By applying (3) for $s = 3$, it follows that $n_2(G) > \frac{17}{24}|G| - \frac{3}{2}$. On the other hand, Theorem D, 1), shows that $n_2(G) \leq \frac{2}{3}|G| - 1$. Thus

$$\frac{17}{24}|G| - \frac{3}{2} < \frac{2}{3}|G| - 1,$$

i.e. $|G| < 12$, contradicting the fact that G is not supersolvable.

If $p = 3$, then the first four element orders of G are 1, 2, 3, 4, and the inequality (3) for $s = 4$ leads to

$$n_3(G) > \frac{17}{12}|G| - 3 - 2n_2(G).$$

But $n_3(G) = n_3(N) = 3^r - 1$ and therefore

$$n_2(G) > \frac{17}{24}|G| - \frac{3^r}{2} - 1.$$

So, we have

$$\frac{17}{24}|G| - \frac{3^r}{2} - 1 < \frac{2}{3}|G| - 1,$$

i.e. $q \in \{1,2,3\}$, and the condition $o(M) < \frac{31}{12}$ implies that $M = D_8$. It is now easy to check that $\pi_e(G) = \{1,2,3,4\}$ and $n_1(G) = 1$, $n_2(G) = 5 \cdot 3^r$, $n_3(G) = 3^r - 1$, $n_4(G) = 2 \cdot 3^r$. Thus

$$o(G) = \frac{1 + 10 \cdot 3^r + 3(3^r - 1) + 8 \cdot 3^r}{8 \cdot 3^r} = \frac{21 \cdot 3^r - 2}{8 \cdot 3^r} > \frac{31}{12},$$

a contradiction.

Subcase 2.2. M is not a 2-group

Then the structure of M is given by Lemma 2.2, d), namely $M \cong K \rtimes H$ is a Frobenius group with kernel $K \cong C_3^m$ and complement $H \cong C_2$. Also, we observe that $p \neq 3$. Indeed, if $p = 3$, then G is strictly 2-closed and consequently supersolvable, a contradiction.

Assume first that $p \geq 5$. The inequality (3) for $s = 3$ becomes

$$n_2(G) > \frac{5}{12}\,|G| - 2.$$

On the other hand, $n_2(G)$ is the number of Sylow 2-subgroups of G and therefore

$$n_2(G) \mid p^r 3^m = \frac{|G|}{2}.$$

If $n_2(G) \neq \frac{|G|}{2}$, then $n_2(G) \leq \frac{|G|}{6}$, implying that

$$\frac{5}{12}\,|G| - 2 < \frac{|G|}{6}.$$

This leads to $|G| < 8$, a contradiction. Thus $n_2(G) = \frac{|G|}{2}$.

As in the proof of Lemma 2.2, d), $G' = NK$ is the unique subgroup of index 2 in G. Clearly, all 3-elements of G are contained in G' and so we have

$$n_3(G) = n_3(G') \leq \frac{3}{4}\,|G'| - 1 = \frac{3}{8}\,|G| - 1$$

by Theorem D, 1). Since the first four element orders of G are 1, 2, 3 and p, from (3) with $s = 4$ we obtain

$$\begin{aligned} n_3(G) &> \frac{p - \frac{31}{12}}{p-3}\,|G| - \frac{p-1}{p-3} - \frac{p-2}{p-3}\,n_2(G) \\ &= \frac{6p - 19}{12(p-3)}\,|G| - \frac{p-1}{p-3}. \end{aligned}$$

Thus

$$\frac{6p-19}{12(p-3)}\,|G|-\frac{p-1}{p-3}<\frac{3}{8}\,|G|-1,$$

which is equivalent to $|G|<12$, a contradiction.

Assume next that $p=2$. Then G possesses exactly 3^m Sylow 2-subgroups. If $S\cong C_2^r\rtimes C_2$ is one of them, then

$$s_1(S)\le s_1(D_8\times C_2^{r-2})=3\cdot 2^{r-1}-1$$

by Theorem D, 5), and so S has at least

$$2^{r+1}-1-(3\cdot 2^{r-1}-1)=2^{r-1}$$

elements of order ≥ 4. This implies that G has at least $2^{r-1}3^m$ elements of order ≥ 4 and at most

$$2^{r+1}3^m-1-2^r(3^m-1)-2^{r-1}3^m=2^{r-1}3^m+2^r-1$$

elements of order 2. Similarly with the case $p\ge 5$, we get

$$n_3(G)\le\frac{3}{8}\,|G|-1$$

and since the first four element orders of G are 1, 2, 3, 4, from (3) with $s=4$ we get

$$n_3(G)>\frac{17}{12}\,|G|-3-2n_2(G).$$

Thus

$$\frac{17}{12}\,|G|-3-2n_2(G)<\frac{3}{8}\,|G|-1,$$

implying that

$$n_2(G)>\frac{25}{48}\,|G|-1.$$

Consequently, one obtains

$$\frac{25}{48}\,|G|-1<2^{r-1}3^m+2^r-1,$$

which means $3^m<\frac{24}{13}$, a contradiction.[1]

Finally, we prove that if for a finite group G we have $o(G)=\frac{31}{12}$, then $G\cong A_4$. We observe first that G is solvable and all its proper quotients are supersolvable. Also, the

[1] Note that the smallest example of such a finite group is S_4.

equality $12\psi(G) = 31|G|$ implies that both 2 and 3 divide $|G|$. Let $1 = d_1 < d_2 < \ldots < d_r$ be the element orders of G. Then

$$\psi(G) = \sum_{i=1}^{r} d_i n_{d_i}(G) = \sum_{i=1}^{r} d_i \varphi(d_i) n'_{d_i}(G),$$

where $n'_{d_i}(G)$ denotes the number of cyclic subgroups of order d_i in G, $\forall\, i = 1, 2, \ldots, r$. Since $\varphi(d_i)$ is even for all $d_i > 2$, we infer that $\psi(G) \equiv 1 \pmod 2$. This shows that $|G| = 4n$, where n is odd and divisible by 3.

If G is supersolvable, then we get $G \cong G' \rtimes C_2^2$, as in the proof of Lemma 2.2, d). It follows that G has a quotient G_1 of order 12. Then $o(G_1) \leq \frac{31}{12}$, contradicting the fact that $o(X) > \frac{31}{12}$ for all supersolvable groups X of order 12.

If G is not supersolvable, then it is a just non-supersolvable group and therefore $G \cong N \rtimes M$ with M and N as above. It is easy to see that the unique possibility to have $o(G) = \frac{31}{12}$ appears in **Case 1** for $r = 2$, i.e. for $G \cong C_2^2 \rtimes C_3 \cong A_4$, as desired. □

Proof of Theorem 1.2. It follows immediately from Theorem 1.1 and Corollary 2.3. □

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